Visualizing the Topology of Vector Fields – An Annotated Bibliography

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Primary Sources

[1] H. Poincaré, Sur les courbes définés par une équation différentielle. *J. Math.* **1**, 167-244 (1875); **2**, 151-217 (1876); **7**, 375-422 (1881); **8**, 251-296 (1882); "Les méthodes nouvelles de la mécanique céleste," *Œuvres*, Vol 1. Gauthier-Villars, Paris, 1892; Paris, 1928. Gauthier-Villars.

Poincaré was the first to understand and document this beautiful mathematics. This, which was only one of his major contributions, essentially founded the field known today as "dynamical systems theory". He discovered it while studying the solutions to the systems of ODEs that arise in celestial mechanics, but since solutions to ODEs are equivalent to flows and hence to vector fields the theory has many applications. Poincare understood the necessity for, and wrote about, the existence of what we now call "strange attractors", "chaos", and "fractals".

[2] A.A. Andronov, E.A. Leontovich, I.I. Gordon, and A.G. Maier, *Qualitative Theory of Second-Order Dynamic Systems*, Translated from Russian by D. Louvish, Halsted Press, John Wiley and Sons, New York, Toronto; Israel Program for Scientific Translations, Jerusalem, London.

[3] A.A. Andronov, E.A. Leontovich, I.I. Gordon, and A.G. Maier, "Theory of Bifurcations of Dynamic Systems on a Plane (Teoriya bifurkatsii dinamicheskikh sistem na ploskosti)," Izdatel'-stovo "Nauka", Glavnaya Redaktsiya, Fizido-Matematicheskoi Literatury, Moskva (1967). Translated from Russian, Israel Program for Scientific Translations, Jerusalem (1971).

Andronov's work is a very complete theory of the topology of two-dimensional vector fields in the plane. His work climaxes with what is essentially an algorithm to determine whether or not two such vector fields are topologically equivalent. His work is quite readable and contains many beautiful examples.

[4] Smale, S., "Differentiable dynamical systems", *Bull. Amer. Math. Soc*. 73 (1967), 747-817.

This is a seminal paper, opening up a new chapter in the application of topology to the study of dynamical systems, and engendering a large quantity of important research in its wake. Readable by anyone familiar with the definition of smooth manifolds and diffeomorphisms.

[5] Asimov, D., "Homotopy of non-singular vector fields to structurally stable ones," *Annals of Math*., Ser 102, (1975) 55-65.

Shows that each non-singular vector field can be continuously deformed, through other such, to a structurally stable one. Aimed at research topologists.

Modern Introductory and Intermediate Surveys

[6] R.H. Abraham and C.D. Shaw, *Dynamics: The Geometry of Behavior*, parts 1-4, Ariel Press, Santa Cruz, CA. (1984).

Everyone should read all four volumes of Abraham and Shaw. It will only take about an hour per volume since they are in the style of comic books. The drawings are fantastic, covering almost all of every page, and the text is both amusing and clear. These have recently been re-issued as a single bound volume. I can only hope that the authors continue their collaboration.

[7] M. Braun, *Differential Equations and Their Applications*, Springer-Verlag, New York (1978).

[8] M. Hirsch and S. Smale, *Differential Equations, Dynamical Systems and Linear Algebra*, Academic Press, New York (1974).

Both of these are excellent, basic texts on ordinary differential equations that emphasize the relationship between the geometric and algebraic approaches to the subject. Braun is a little more traditional.

[9] L. Perko. *Differential Equations and Dynamical Systems.* Springer-Verlag. (1991)

A more modern (trendy?) approach than either of the above, this text is much more geometrical and contains an introduction to bifurcation theory and chaos.

[10] V.I. Arnold, *Ordinary Differential Equations*, MIT Press, (1973).

To understand vector field topology you need to review the geometric (or "qualitative") theory of ordinary differential equations. There are many books on ODEs from a geometrical point of view. The above three are some of my favorites. Braun is more elementary than the other two. But you should look at Arnold, at least for the pictures. It is a good idea to look at anything written by Arnold. He is the modern master of the geometric approach to just about anything.

[11] Irwin, M.C., *Smooth Dynamical Systems*, Academic Press, 1980.

This excellent textbook clearly lays out many of the important developments in the aftermath of Smale's "Differentiable dynamical systems" paper. Suitable for a graduate student in mathematics or possibly an advanced undergraduate with a strong math background.

[12] J. Hale and H. Koçak. *Dynamics and Bifurcations.* Springer-Verlag. (1991)

Hale and Koçak have produced an introduction to bifurcation theory that is complete and yet assumes a minimum of mathematical background (an undergraduate course in ODEs and one in linear algebra should suffice). The bibliographical notes and non-trivial examples alone are worthwhile.

[13] J. Guckenheimer and P. Holmes. *Nonlinear Oscillations, Dynamical Systems, and Bifurcations of Vector Fields,* Springer Verlag (1983).

Their goal was to start for three-dimensions what Andronov completed for two dimen-

sions. This excellent book is a pleasure to read and contains an introduction to many of the modern topics in dynamical systems theory: stability, chaos, symbolic dynamics, bifurcation theory, higher-order singularities, catastrophe theory, and physical applications. If you read one book (after Abraham and Shaw) this should be it.

[14] S. Wiggins. *Introduction to Applied Nonlinear Dynamical Systems and Chaos*. Springer-Verlag. (1990).

Longer, more pedagogical and more introductory than Guckenheimer and Holmes (above), this volume covers most of the same material. It is more suitable for those starting from a less mathematical background, and who want a more complete text. Wiggins has written several other books, all of which are worth investigating.

Applications Of Vector Field Topology To Fluid Dynamics

[15] M.J. Lighthill, "Attachment and Separation in Three Dimensional Flow," *Laminar Boundary Layers II*, ed. L. Rosenhead, pp. 72-82, Oxford University Press (1963).

This appears to be the first paper where the qualitative theory of differential equations is applied in the "physical plane" of fluid flow rather than the phase plane of solutions of a system of ODEs.

[16] A.E. Perry and B.D. Fairly, "Critical Points in Flow Patterns," *Advances in Geophysics* 18 B pp. 299-315 (1974).

Another early paper classifying flow patterns using Poincare's theory. Both of these are worth digging up if you are interested in fluid dynamics applications. They provide readable historical context.

[17] M.S. Chong. A.E. Perry, B.J. Cantwell, "A General Classification of Three-Dimensional Flow Fields," *Phys. Fluids* A **2** (5) pp. 765-777, May (1980).

Perry, Chong, and Cantwell have written many papers in this area, all interesting. This is a summary of their more elementary work that characterizes three dimensional singular points in flows based on the first three matrix invariants (trace, determinant, and determinant of cofactors). This in contrast to classifying singular points based on the eigenvalue distribution, though it is formally equivalent.

[18] A.E. Perry, "A Study of Degenerate and Non-degenerate Critical Points in Three-dimensional Flow Fields," Deutsche Forschungs- und Versuchsanstalt fur Luft und Raumsfahrt report DFVLR-FB 84-36.

Here the constraints of the Navier-Stokes equations, and in some cases symmetry constraints as well, are analyzed as to their effect on stabilizing critical points that would normally be considered unstable in an arbitrary vector field. Among other things, it provides a way to check on the validity of a numerical solution based on the types of critical points that are found.

[19] U. Dallman, "Topological Structures in Three-dimensional Vortex Flow Separation," *AIAA 16th Fluid and Plasma Dynamics Conference*. 1983. Paper AIAA-83-1735.

[20] U. Dallman, “Structural Stability of Three-Dimensional Vortex Flows,” in *Nonlinear Dynamics of Transcritical Flows: Proceedings of a DFVLR International Colloqium.* Bonn, Germany, 1984. Edited by H.L. Jordan. Lecture Notes in Engineering Vol 13. Springer-Verlag

Dallman also uses the physics constraints of the Euler and Navier-Stokes equations to derive topological possibilities for fluid flows. His papers have the best pictures of complicated flow patters that I have seen. That so much complex three-dimensional information can be conveyed by simple black-and-white line drawings is both humbling and inspirational.

[21] J.C.R. Hunt, et al. “Kinematical Studies of the Flow Around Free or Surface-Mounted Obstacles; Applying Topology to Flow Visualization,” *J. Fluid Mechanics* Vol 86, part 1 pp. 179-200. 1978

This paper is one of several that uses index theorems to analyze patterns of multiple critical points in complicated flow fields. The drawings are interesting, although not as nice as those of Dallman.

Vortices and Vortex Topology

[22] A.E. Perry and H. Hornung, “Some Aspects of Three-Dimensional Separation, Part II: Vortex Skeletons,” *Z. Flugwiss, Weltraumforsch.* **8**, Heft 3, pp. 155-160 (1984).

The “vortex skeleton” approach outlined here is a powerful technique for deriving (and drawing) simple topological structures that explain the global dynamics of vector fields. This paper makes use of interesting magnetic field experiments (where the current-carrying wires are the vortex skeleton) to shed light on fluid flows.

[23] U. Dallman, “Three-dimensional Vortex Structures and Vorticity Topology,” *Fluid Dynamics Research* **3** pp. 183-189 (1988).

Another beautiful paper by Dallman, this time dealing with the topology of the vorticity vector field (which, unlike the velocity field, is invariant under Gallilean transformations).

[24] Y. Levy and A. Seginer, “Graphical Representation of Three-Dimensional Vortical Flows by Means of Helicity Density and Normalized Helicity,” *AIAA 6th Applied Aerodynamics Conference*, 1988, Williamsburg VA. paper AIAA-88-2598.

Helicity, $\vec{V} \bullet (\nabla \times \vec{V})$, is an important quantity in magnetohydrodynamics and in the more advanced aspects of topological fluid mechanics not touched on in this tutorial. In this paper, Levy and Seginer show that helicity contours can be effectively used to visualize vortex cores.

[25] L.A. Yates and G.T. Chapman, “Streamlines, Vorticity Lines, and Vortices,” American Institute of Aeronautics and Astronautics, paper AIAA-91-0731.

Here, multiple definitions of “vortex core” are compared and contrasted. Vortex cores defined via critical points, via helicity, via streamline curvature, and via integral curves through the vorticity field are computed and situations where they differ are emphasized. This paper is an important step in the formalization of the concept of a “vortex” for three-dimensional steady flows.

Topological Analysis Of Specific Flows

[26] U. Dallman and G. Schewe, “On the Topological Changes of Separating Flow Structures at Transition Reynolds Numbers,” American Institute of Aeronautics and Astronautics, paper AIAA-87-1266.

[27] U. Dallman and B. Schulte-Werning. “Topological Changes of Axisymmetric and Non-axisymmetric Vortex Flows,” Proc. *IUTAM Symp. on Topological Fluid Mechanics.* 1989. H.K. Moffatt and A. Tsinober, editors. Cambridge University Press.

> Both of the above papers deal with the changes in the topology of a flow field as some parameter, such as reynolds number, angle of attack, or geometry is changed. They are worthwhile for the pictures alone.

[28] A.E. Perry and J.H. Watmuff, “The Phase-Averaged Large-Scale Structures in Three-Dimensional Turbulent Wakes,” *J. Fluid Mech.* **103** pp. 33-51.

[29] A.E. Perry and D.K. Tan, “Simple Three-Dimensional Vortex Motions in Coflowing Jets and Wakes,” *J. Fluid Mech.* **141** pp. 197-231, (1984).

> In these papers, Perry et al apply topological analysis to vector fields obtained from wind and water tunnel experiments. They work out the topological structures in (cross-sections of) wakes and jets.

[30] S. Ying, L.Schiff and J.L. Steger “A Numerical Study of Three-Dimensional Separated Flow Past a Hemisphere Cylinder” *Proc. AIAA 19th Fluid Dynamics, Plasma Dynamics and Lasers Conference.*

> This paper reports on the results of a set of supercomputer simulations for three-dimensional flow about a simple geometry. Most notable in the context of this bibliography are three items: First, as the simulation was refined (using more mesh points) the authors encountered more intricate topological structures. Second, there are very pretty color pictures illustrating the flow topology that were laboriously hand drawn from multiple cross sectional vector plots and streamlines. Finally, this dataset has been used by Hellman and Hesselink, and by Globus, Levit and Lasinski as a test case for their automated topology extraction software.

[31] C.H. Hung, P.G. Buning, “Simulation of Blunt-Fin-Induced Shock-Wave and Turbulent Boundary-Layer Interaction,” *J. Fluid Mech.* (1985), Col. 154, pp. 163-185.

> The “blunt fin” is becoming a classic dataset for fluid flow visualization studies. This is the paper describing the solution, and it contains drawings of the flow topology.

[32] C.M. Hung, C.H. Sung, and C. L. Chen, “Computation of a Saddle Point of Attachment,” Proc. *AIAA 22nd Fluid Dynamics, Plasma Dynamics, and Lasers Conference.* Paper AIAA-91-1713

[33] M. R. Visbal. “Structure of Laminar Juncture Flows,” Proc. *AIAA 20th Fluid Dynamics, Plasma Dynamics, and Lasers Conference.* Paper AIAA 89-1873

> Both of these papers deal with the topology of the horseshoe vortex that forms in the flow past a cylinder protruding from a flat plate. There are some surprises here, and it is found

that attachment and separation points can have a complicated structure. The second paper also deals with the transition to unsteady flow in this geometry.

[34] K.C. Wang et al., “Three-Dimensional Separated Flow Structure over Prolate Spheroids,” *Proc. R. Soc. Lond*. A 421, pp 73-90, 1990.

The authors present a very nice set of flow visualization experiments illustrating the topology of surface flow patterns on prolate spheroids as the angle of attack is varied.

Software Implementations Of Topological Techniques

[35] L. Hesselink and J. Helman, “Evaluation of Flow Topology from Numerical Data,” American Institute of Aeronautics and Astronautics, paper AIAA-87-1181.

[36] J. L. Helman and L. Hesselink, “Surface Representation of Two- and Three-Dimensional Fluid Flow Topology,” *Proc. Visualization ‘90,* San Francisco, IEEE Computer Society Press. (1990).

[37] J. L. Helman and L. Hesselink, “Analysis and Representation of Complex Structures in Separated Flows,” S*PIE Conf on Extracting Meaning From Complex Data*, San Jose, (1991).

[38] J.L. Helman and L. Hesselink, “Representation and Display of Vector Field Topology in Fluid Flow Data Sets,” *IEEE Computer*, pp. 27-36, Aug. 1989. Also appears in *Visualization in Scientific Computing*, G. M. Fielson & B. Shriver, eds. Companion videotape available from IEEE Computer Society Press.

Jim Helman’s Ph.D. research was on the computerized extraction of vector field topology. He implemented systems for both two and three dimensions which found critical points and integrated both singular streamlines and singular streamsurfaces.

[39] S. Shirayama and K. Kuwahara, “Flow Past a Sphere: Topological Transitions of the Vorticity Field,” American Institute of Aeronautics and Astronautics, paper AIAA-90-3105-CP.

As an almost incidental part of this study of vorticity topology of flow past a sphere, the authors implemented critical point finding and classification algorithms.

[40] A. Globus, C. Levit, T. Lasinski, “A Tool for Visualizing the Topology of Three-Dimensional Vector Fields,” Proc. *Visualization ‘91*. San Diego, 1991. IEEE Computer Society Press.

This paper describes the first implementation (to the author’s knowledge) of vector field topology analysis software in an integrated production scientific visualization environment. The purpose the paper is to provide a guide to others who wish to do the same.

[41] R. R. Dickinson, “Interactive Analysis of the Topology of 4D Vector Fields,” *IBM J. Res. Develop.*, v. 35 no. 1/2, pp 59-66. January/March 1991.

Dickinson describes software he implemented as part of the “Visual Edge” software system. The user moves a cursor in 2D or 3D space and the nearest saddle point is located in real time. Special curves emanating from this saddle point are drawn. Objects can be saved as part of a composite visualization. The 4D vector fields referred to in the title are a collection of 3D vector fields that are ordered in time.

[42] K. M. Yip, “Extracting Qualitative Dynamics from Numerical Experiments,” MIT AI Memo 950. Massachusetts Inst. of Technology, 1987.

While not specifically tailored to vector field topology, the “dynamicist’s workbench” described herein is an innovative system for studying the structure of multi-parameter numerical solution spaces. It has some general capabilities of interest for advanced systems. This software has vision.

Software Systems for Vector Field Visualization.

[43] G. Bancroft, F. Merritt, T. Plessel, P. Kelaita, R. McCabe, A. Globus, “FAST: A Multi-Processing Environment for Visualization of CFD,” *Proc.Visualization ‘90*, IEEE Computer Society, San Francisco (1990).

Fast is an integrated flow visualization environment which supports vector field topology analysis as well as more traditional visualization techniques.

[44] P. P. Walatka, P. G. Buning, *PLOT3D User’s Manual,* NASA Technical Memorandum 101067, NASA Ames Research Center.

Plot3d is the classic flow visualization system which is still used for most aerodynamic applications. Competing systems should strive to have all of its advantages and few of its flaws.

[45] C. Levit and S. Bryson, “A Virtual Environment for Exploration of Three Dimensional Flowfields,” SPIE paper 1457-19 *SPIE Conf on Stereoscopic Displays and Applications II*. San Jose (1991).

Virtual environments for exploring vector fields are an interactive alternative to the more “batch”-oriented techniques of topological analysis. An ideal system would combine both techniques.

[46] J.P.M. Hultquist, “Interactive Numerical Flow Visualization Using Stream Surfaces,” *Computing Systems in Engineering* **1** (2-4) pp. 349-353.

Singular streamsurfaces, and general streamsurfaces combined with topological information, are powerful visualization tools for flow fields and other vector fields as well. The visualization system by Hultquist allows interactive placement of initial curves for streamsurfaces, and real-time generation of streamsurfaces.

Relevant Numerical Techniques

[47] R.A. Walker, “Computing the Jordan Form for Control of Dynamic Systems,” Guidance and Control Laboratory, Department of Aeronautics and Astronautics, Stanford University (1981).

[48] G.H. Golub and J.H. Wilkinson, “Ill-conditioned Eigensystems and the Computation of the Jordan Canonical Form,” *SIAM Review* Vol. 18 No. 4 pp578-619. October 1976.

These last two papers deal with the obscure topic of accurate numerical computation of non-trivial Jordan blocks. This is used when finding generalized eigenvectors corresponding to eigenvalues with geometric multiplicity - useful when degenerate critical points are

likely to be found.

Books on Topological Fluid Mechanics

[49] H.K. Moffatt and T. Tsinbor, *Topological Fluid Mechanics - Proceedings of the IUTAM Symposium*, Cambridge University Press, 1989.

> This book contains a wealth of papers on fluid and magnetohydrodynamic flow physics from a topological point of view. While not specifically oriented to visualization, the papers are well illustrated and extremely interesting. There is a lot of scientific visualization software that is waiting to be written based on the papers in this book.

[50] H.N. Shirer and R. Wells, *Mathematical Structure of the Singularities at the Transitions Between Steady States in Hydrodynamic Systems*, Lecture Notes in Physics v. 185, 1983, Springer-Verlag.

> Catastrophe theory is an alternative approach to analyzing singularities of mappings (flows). This rather technical volume treats transitions between stable fluid flow topologies using this technique.

[51] P.G. Bakker, *Bifurcations in Flow Patterns: Some Applications of the Qualitative Theory of Differential Equations*, 1991, Kluwer Academic Publishers.

> This excellent (though expensive) volume deals with the local and global topological structures that are possible within physically realistic flow fields. It is the first full-length book that I know of covering this subject. It extends the work of Perry and Dallman to the case of transitional structures.